ELSEVIER

Discrete Applied Mathematics 69 (1996) 257–270

DISCRETE APPLIED MATHEMATICS

The complexity of generalized graph colorings

Jason I. Brown

Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 355

Received 1 November 1994; revised 23 June 1995

Abstract

Given a graph property P and positive integer k, a P k-coloring of a graph G is an assignment of one of k colors to each vertex of the graph so that the subgraphs induced by each color class have property P. This notion generalizes the standard definition of graph coloring, and has been investigated for many properties. We consider here the complexity of the decision problem. In particular, for the property $-G$, of not containing an induced subgraph isomorphic to G, we conjecture (and provide strong evidence) that $-G$ k-colorability is NP-complete whenever G has order at least 3 and $k \geqslant 2$. The techniques rely on new NP-completeness results for hypergraph colorings.

1. Introduction

Many generalizations of graph colorings have been investigated in the literature. Most of them can be incorporated into the following framework. Let P be a *property* of graphs, i.e. a collection of graphs containing K_0 and K_1 that is closed under isomorphism (for example, acyclicity, planarity, perfection, completeness and independence are all properties). Given a graph G and a positive integer k, a P *k-coloring* of G is a function $\pi : V(G) \to \{1,\dots,k\}$ such that the subgraph induced by each color class $\pi^{-1}(i)$ has property P. The P *chromatic number* of G, $\chi(G : P)$ is the least k for which G is P k-colorable. When P is the property of being edge–free, P colorings and standard graph colorings coincide. Such a broad generalized notion of graph coloring can be found in [26,17,24], and the development of the theory of P colorings can be found in [7–9]. In fact, it seems that P colorings are a useful way to classify hierarchically all graphs with respect to a property P (see [10]). For other works on the topic, see [1,2,5,6,11,21,31,37–39]; [11] applies P colorings to give a graphical approach to hypergraph colorings, and shows the utility of considering P colorings even if one is interested only in hypergraph colorings.

We remark that generalized graph colorings may be placed in a Ramsey–like context: write $P \to^k H$ if for every assignment of one of the colors $1,\dots,k$ to the vertices of H, there is a monochromatic subgraph with property P. For the properties P of

SSDI 0166-218X(96)00096-8

not containing an induced subgraph isomorphic to a fixed graph G, this problem originated with Folkman [20], and numerous existential and extremal problems have been investigated [13, 14, 20, 25, 28, 32–34].

Let us return to standard graph colorings for the moment. In addition to proving theoretical results in chromatic theory, it is natural to investigate colorings from an algorithmic perspective, for graph colorings arise in a myriad of applications, and there are many algorithms (both exact and heuristic) for coloring the vertices of a graph. One of the first problems to have its complexity classified (see [22]) was the following (for fixed k):

Gra(k) (Graph k-colorability)

Instance: A graph F.

Question: Is F k-colorable?

Gra(k) is polynomial for $k = 1$ and 2, while it is NP-complete for $k \geqslant 3$. The difficulty in k-coloring a graph for $k \geqslant 3$ seems tied closely to the problem of classifying all $(k+1)$-critical graphs (i.e. k-chromatic graphs all of whose proper subgraphs are $(k-1)$-colorable.

We will investigate here the complexity of P colorability. While it is unlikely to find a complete solution to the problem, we will center our discussion on properties $P = -G$ (i.e. not containing an induced copy of G). Along the way, we will need to prove some new NP-completeness results for hypergraph colorings which are of interest in their own right.

We note that [23] considers the complexity of coloring graphs where each color class is either complete or independent. The complexity of quite a different notion of generalized colorings related to graph homomorphism has been solved in [27]. Finally, Burr [16] has investigated the complexity of edge colorings related to Ramsey theory.

For graph and hypergraph theoretic notation, we refer the reader to [4,3] and for complexity terminology, to [22]. The *order* of a graph is its number of vertices (if G has order at least 1, then we say that G is *nonempty*). The complete, path and cycle graphs of order n are denoted by K_n, P_n and C_n respectively. The clique number of G is denoted by $\omega(G)$. We often do not distinguish between a subset of $V(G)$ and the subgraph of G it induces.

The complement of a graph G is written as $\overline{G}$. Given two disjoint graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and $G+H$, the *join* of G and H, is the graph formed from $G \cup H$ by adding in all edges between G and H. Given a vertex v, the graph formed by *substituting H for vertex v in G* arises from $(G-v) \cup H$ by making every vertex of H adjacent to every neighbor of v in G. The chromatic number of a graph is denoted by $\chi(G)$.

We assume that all properties under discussion are nontrivial (i.e. do not contain all graphs) and *hereditary*, that is, closed under induced subgraphs (this implies that the restriction of any P coloring to an induced subgraph is again a P coloring, a desirable general condition). For any such property, there are graphs of all P chromatic numbers

[9,8]. The *elementary* properties are those of the form $-G$, which (for a fixed graph G of order at least 2) denotes the family of G-free graphs (i.e. those that do not contain an induced copy of G); these are elementary properties as any hereditary property is the intersection of (possibly infinitely many) such properties. A graph is called a P *graph* if it belongs to P.

One construction of P k-chromatic graphs is via the substitution operation as follows. We say that G is *vertex P k-critical* if $\chi(G:P)=k$ but for all vertices v of G, $G-v$ is P $(k-1)$-colorable (so in particular, a vertex P 2-critical graph is a graph minimally not in P). If G is a vertex P 2-critical graph and H is a P k-chromatic graph, then the graph formed from G by substituting a copy of H for every vertex but one of G is not P k-colorable (see [9]). One of the earliest results on generalized graph colorings arose from J. Folkman's work in Ramsey theory, and he proved the following:

Theorem 1.1 (Folkman's Theorem [20]). *For any graph G of order at least 2 and any positive integer k there is a graph H that is not $-G$ k-colorable, and moreover, $\omega(H)=\omega(G)$.*

This result is a broad generalization of the existence of triangle–free graphs of arbitrarily large chromatic number.

Finally, we refer the reader to [22] for complexity terminology. The class Poly denotes the decision polynomials that have polynomial time algorithms, and NP is the class of nondeterministically polynomial decision problems. NPc is the class of NP-complete problems.

In the next section, we prove some new results on hypergraph colorings that will not only be useful to us later, but are of independent interest in their own right. Section 3 investigates the complexity of P k-colorability, which can range from polynomial times to undecidable. We formulate the conjecture that $-G$ k-colorability is NP-complete for any graph G of order at least 3 and any $k \geqslant 2$ (contrast this with the well known polynomial time algorithm for graph 2-colorability), and provide evidence for many graphs G. In the final section, we provide general NP-hardness results for P k-colorabililty, and examine the extension of the problems to countable (but finitary) graphs.

2. Hypergraph colorability

In our approach to the problem of P k-colorability, we will sometimes consider hypergraphs. We spend this section reviewing known results for hypergraph colorability, and proving new NP-completeness results that will be useful in P colorings.

We refer the reader to [3] for all standard terminology regarding hypergraphs. For a hypergraph H, we denote its vertex set by $V(H)$ and edge set by $E(H)$; the *order* and *size* of H are respectively $|V(H)|$ and $|E(H)|$. A hypergraph H is r-uniform if every edge of H has size r (all hypergraphs considered here will be *loopless*, i.e. all edges have cardinality at least 2). A *path* of length $k \geqslant 1$ in hypergraph H is an alternating sequence $v_1, E_1, v_2, E_2, \ldots, v_{k+1}$ of distinct vertices and edges such that $v_i,\ v_{i+1} \in E_i$ for $i = 1, \ldots, k$. A *cycle* of length $k \geqslant 2$ in hypergraph H is an alternating sequence $v_1, E_1, v_2, E_2, \ldots, v_k E_k$ of distinct vertices and edges such that $v_1, E_1, v_2, E_2, \ldots, v_k$ is a path and $v_1,\ v_k \in E_k$; the *girth* of H, $\gamma(H)$, is the length of a shortest cycle (if H has no cycles, then we define $\gamma(H) = \infty$).

An independent set I of hypergraph H is a subset of $V(H)$ that contains no edge of H. A hypergraph H is *k-colorable* if there is a function $\pi : V(H) \to \{1, \ldots, k\}$ such that no edge is monochromatic, i.e. for any edge e of H, $|\pi(e)| \geqslant 2$ (it is easy to see that H is k-colorable if and only if $V(H)$ can be covered by k independent sets). The chromatic number of H, $\chi(H)$, is the least k for which H is k-colorable. Finally, a hypergraph is *k-critical* if $\chi(H) = k$ but the deletion of any vertex or edge leaves a $(k-1)$-colorable hypergraph.

We remark that there is another notion of hypergraph coloring, where no colour is repeated in any edge; such a coloring is called a strong coloring, and there is a somewhat smaller literature on such colorings [3].

Let $r \geqslant 2$ and $k \geqslant 1$. Consider the following problem:

Hyp(r,k) (r-uniform Hypergraph k-colorability)

Instance: An r-uniform hypergraph H.

Question: Is H k-colorable?

For $r = 2$, Hyp(r,k) is Gra(k), graph k-colorabililty, and as mentioned above, it is known to be polynomial if $k \leqslant 2$ and to be NP-complete if $k \geqslant 3$. For $r \geqslant 3$ and $k \geqslant 3$, Hyp(r,k) is NP-complete by the work of Phelps and Rödl [35] (even when restricted to hypergraphs without 2-cycles). Also, in Section 4 of [11] a polynomial transformation from Gra(k) to Hyp(r,k) that preserves chromatic number was constructed, and the NP-completeness of Hyp(r,k) follows from that of Gra(k) (we point out that this transformation utilizes generalized graph colorings). Lovász [30] provided a polynomial transformation from Gra(k) to the problem

Given a hypergraph, is it 2-colorable?

This result is surprising, for remember that graph 2-colorability is polynomial. The hypergraphs formed in Lovász's construction are only uniform for $k = 3$, however. We can extend his argument to prove Hyp($r,2$) is NP-complete for any $r \geqslant 3$ (we will need this result in the next section).

Lemma 2.1. Hyp($r,2$) *is* NP-*complete for all fixed* $r \geqslant 3$.

Proof. We reduce Hyp($r-1,r$) to Hyp($r,2$) (for $r = 3$, this is Lovász's argument). Given an $(r-1)$-uniform hypergraph H with vertex set $\{v_1, \ldots, v_n\}$, we take (disjoint) copies $H^1, \ldots, H^r$ of H. We form an r-uniform hypergraph H' on vertex set $V(H^1) \cup \cdots \cup V(H^r) \cup \{\infty\}$ (where ∞ is a new vertex) by taking edges of the form

(i) $e^i \cup \{\infty\}$, where $e^i \in E(H^i)$, $i = 1, \ldots, r$, and

(ii) $\{v_j^1, \ldots, v_j^r\}$, where $j = 1, \ldots, n$.

The transformation $H \to H'$ is clearly polynomial.

Suppose $\pi : V(H') \to \{1,2\}$ is a 2-coloring of H'. Consider the sets $C^i = \{v_j \in V(H) : \pi(v_j^i) = \pi(\infty)\}$ for $i = 1, \ldots, r$. Every $v_j \in V(H)$ is in some C^i as no edge of type (ii) is monochromatic under π. Also, no C^i contains an edge of H, as no edge of type (i) is monochromatic under π. Thus $C^1, \ldots, C^r$ is a set of r independent sets of H that cover $V(H)$, and hence H is r-colorable.

Conversely, if $\pi : V(H) \to \{1, \ldots, r\}$ is an r-coloring of H, then one can easily verify that $\pi' : V(H') \to \{1,2\}$ such that

$$\pi'(v_j^i) = 1 \text{ if and only if } \pi(v_j) = i$$
$$\pi'(\infty) = 1$$

is a 2-coloring of H'.

Thus we have polynomially reduced Hyp$(r-1, r)$ to Hyp$(r,2)$. As noted earlier, Hyp$(r-1, r)$ is known to be NP-complete for all $r \geqslant 3$, so the NP-completeness of Hyp$(r,2)$ for $r \geqslant 3$ follows. □

We summarize the result as follows.

Theorem 2.2. *For all fixed $r \geqslant 2$ and $k \geqslant 1$,*

- Hyp(r,k) $\in$ Poly *if $k = 1$, or $k = 2$ and $r = 2$, and*
- Hyp(r,k) $\in$ NPc *if $k \geqslant 3$, or $k = 2$ and $r \geqslant 3$.*

We shall also need a slightly stronger result on the following problem.

Hyp(r,k,g)

Instance: An r-uniform hypergraph H of girth greater than g.

Question: Is H k-colorable?

Theorem 2.3. *For all fixed $r \geqslant 3$, $k \geqslant 2$ and $g \geqslant 2$, the problem* Hyp(r,k,g) *is* NP-*complete.*

Proof. For $k \geqslant 3$ this was shown by Phelps and Rödl [35], who transformed Gra(k) into Hyp(r,k,g). They also proved Hyp$(r,2,2)$ is NP-complete. (we remark that in [18] it was shown that deciding 2-colorability of Steiner quadruple systems is polynomial). It remains to show that Hyp$(r,2,g)$ is NP-complete for all $g \geqslant 3$. We shall transform Hyp$(r,2)$ into Hyp$(r,2,g)$.

First we will need an r-uniform hypergraph K of girth $> g$ containing distinct vertices $x, x_1, \ldots, x_{r-1}$ such that the following four conditions hold:

(i) K is 2-chromatic and $\{x, x_1, \ldots, x_{r-1}\} \notin E(K)$.

(ii) In any 2-coloring of K, $\{x, x_1, \ldots, x_{r-1}\}$ is not monochromatic.

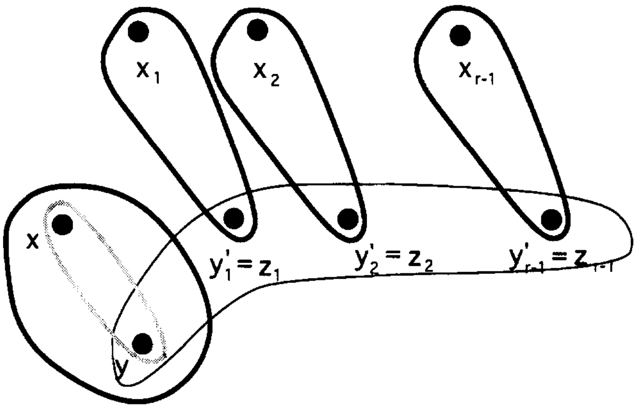

Fig. 1.

(iii) Any 2-coloring of $\{x, x_1, \ldots, x_{r-1}\}$ that is not monochromatic can be extended to a 2-coloring of K.

(iv) No two of $\{x, x_1, \ldots, x_{r-1}\}$ are joined by a path of length $\leqslant g$.

Let K' be any fixed r-uniform 3-critical hypergraph of girth $> g$; the existence of K' follows from the work of Erdös and Hajnal [19] and from [29] on the existence of r-uniform hypergraphs of girth $> g$ with arbitrarily large chromatic number. Note that the order and size of K' is a constant (depending on r and g). Let e be any fixed edge of K' containing vertices x' and y'. Take new vertices $z_1, \ldots, z_{r-1}$ (not in K') and form K'' from $K' - e$ by adding vertices $z_1, \ldots, z_{r-1}$ and the edge $\{y', z_1, \ldots, z_{r-1}\}$. K'' is clearly r-uniform, of girth $> g$ and 2-chromatic (as $K' - e$ is). K'' satisfies (ii) with $x = x'$ and $x_i = z_i$, since in any 2-coloring of $K' - e$, x' and y' must receive the same color; (iii) is satisfied by taking any 2-coloring of $K' - e$ that agrees with the given coloring at x'. However, (iv) is clearly not satisfied since $\{y', z_1, \ldots, z_{r-1}\}$ is an edge of K'. Note that in $K' - e$, x' and y' are not joined by a path of length $\leqslant g - 1$ since K' has girth $> g$. We now take copies $K'_1, \ldots, K'_{r-1}$ of $K' - e$ together with K'' and form the r-uniform hypergraph K of girth $> g$ by identifying y'_i with z_i for $i = 1, \ldots, r-1$; it is not hard to see that (i)–(iii) hold in K with $x_i = x'_i$. Moreover, (iv) holds as well, as by the previous remark on the distance of x' from y' in $K' - e$. Thus K is the required hypergraph (see Fig. 1).

Now given an r-uniform hypergraph, we take $m = |E(H)|$ copies $K^1, \ldots, K^m$ of K and form hypergraph H' from $H, K^1, \ldots, K^m$ by arbitrarily identifying (in a 1–1 fashion) $\{x^i, x^i_1, \ldots, x^i_{r-1}\}$ with the elements of the edge e_i of H, and then removing all edges of H (such a construction was used by Toft [40] to build critical hypergraphs). H' is an r-uniform hypergraph of girth $> g$ since each K^i is an r-uniform hypergraph of girth $> g$ satisfying condition (iv).

We now claim that H is 2-colorable if and only if H' is. If H had a 2-coloring π, then we can extend π to a 2-coloring of each K^i, and hence H', via (iii). Conversely, if π' were a 2-coloring of H', then by (ii), the restriction of π' to H is a 2-coloring of H.

As the construction can clearly be carried out in polynomial time, we have polynomially transformed $\mathrm{Hyp}(r,2)$ into $\mathrm{Hyp}(r,2,g)$, and by Theorem 2.3, we conclude that $\mathrm{Hyp}(r,k,g)$ is NP-complete (all fixed $r \geqslant 3$, $k \geqslant 2$ and $g \geqslant 2$). $\square$

3. The complexity of P-colorings

We now consider the following problem for fixed property P and positive integer k.

Pcol(k) (Graph P k-colorability)
Instance: A graph G.
Question: Is G P k-colorable?

For some choices of (hereditary) properties P, the problem $P\mathrm{col}(k)$ is not even be decidable. For example, if $S \subseteq \{2n+1 : n \geqslant 5\}$ and $P_S = -\{C_m : m \in S\}$, then P is closed under substitution. Let S' and S be two different such sets, and without loss let $m \in S - S'$. Construct inductively a sequence of graphs G_k by setting $G_1 = C_m$, and forming G_{k+1} by substituting G_k for all vertices but one of C_m. As in the proof of Theorem 2.8 of [9], G_k is $P_{S'}$ k-chromatic but belongs to P_S as it is closed under substitution. It follows that for any positive integer k, the uncountably many problems $P_S\mathrm{col}(k)$ are distinct, and hence almost all of them are undecidable!

However, for many of the properties P that are of interest, it will be clear that there is some algorithm for $P\mathrm{col}(k)$, and it is to these problems that we direct our attention. Note that if $P\mathrm{col}(1)$ (i.e. membership in P) is in NP, then $P\mathrm{col}(k) \in \mathrm{NP}$, since if we are given a purported P k-coloring of a graph G, we can use the fact that $P\mathrm{col}(1) \in \mathrm{NP}$ to give a verification that each color class is in P. Also, the complexity of $P\mathrm{col}(k)$ and $\overline{P}\mathrm{col}(k)$ are the same, as a graph G is P k-colorable iff $\overline{G}$ is $\overline{P}$ k-colorable.

There are some properties for which $P\mathrm{col}(k)$ is clearly polynomial. For example, if $m \geqslant 2$ is fixed and P is the collection of all graphs of order at most m, then $P\mathrm{col}(k)$ is clearly polynomial as G is P k-colorable iff G has order at most km. A more interesting example is afforded by the properties $-\{K_m, \overline{K_l}\}$. By Ramsey's theorem, there is an N (depending only on m and l) such that any graph of order at most N contains a K_m or a $\overline{K_l}$. Thus for any graph G of order greater than $k(N-1)$, any subgraph of G of order at least N contains a K_m or $\overline{K_l}$. It follows that no graph of order at least $k(N-1)+1$ is $-\{K_m, \overline{K_l}\}$ k-colorable, and hence a polynomial-time algorithm for $-\{K_m, \overline{K_l}\}$ k-colorability need only check first if the order is at most $k(N-1)$ (a *constant*), and then check for at most N^k possible $-\{K_m, \overline{K_l}\}$ k-colorings whether any of them are indeed a $-\{K_m, \overline{K_l}\}$ k-coloring.

It is interesting to observe that the above shows that there is a polynomial-time algorithm for $-\{K_m, \overline{K_l}\}$ k-colorability even though we do not know it explicitly, as the determination of the precise value of the Ramsey numbers is a very difficult problem. This is similar to Robertson and Seymour's well-known polynomial-time (but undiscovered!) algorithms for recognizing families of minors [36] (though we remark that for our coloring algorithms, an upper bound for N will do as well).

4. G-free k-colorability

One of the most interesting set of properties are those of the form $-G$, where G is a graph of order at least 3. Recall that $-G$ is the property of not containing an induced G. When $G = K_2$, then $-G$ is the property of being independent (i.e. edge-free), and $-G\text{col}(k)$ is the same as $\text{Gra}(k)$. It is obvious that $-G$ 1-colorability is polynomial, as if G has order m, we need only check whether any of the induced subgraphs of order m are isomorphic to G. We state the following conjecture:

Conjecture 4.1. For any graph G of order at least 3, the problem $-G\,\text{col}(k)$ is NP-complete for any $k \geqslant 2$.

We have not been able to prove this conjecture, but have proved a variety of cases for large families of graphs G.

Proposition 4.2. *Let $G = H + K$, where H and K are nonempty. Then $-G\text{col}(k)$ is* NP-*complete for $k \geqslant 3$.*

Proof. As noted above, $-G\text{col}(k) \in \text{NP}$. We reduce $\text{Gra}(k)$ to $-G\text{col}(k)$.

From Folkman's theorem, let F be a fixed graph of such that $\chi(F : -(H \cup K)) > k$ and $\omega(F) = \omega(H \cup K) < \omega(H + K)$. Given a graph W, we construct graph W' by substituting a copy F_v of F for each vertex v of W. The transformation is clearly polynomial. We will show that W' is $-G$ k-colorable if and only if W is k colorable (this generalizes constructions in [9, 11]).

If $\pi : V(W) \to \{1,\dots,k\}$ is a k-coloring of W, then define $\pi' : V(W') \to \{1,\dots,k\}$ such that $\pi'(F_v) = \pi(v)$. Each color class of π' is the disjoint union of copies of F, and hence have clique number $\omega(F) < \omega(H + K)$. It follows that each color class is G-free, and π' is a $-G$ k-coloring of W'.

On the other hand, suppose that $\pi' : V(W') \to \{1,\dots,k\}$ is a $-G$ k-coloring of W'. As $\chi(F : -(H \cup K)) > k$, for every vertex v of W, there is a monochromatic copy of $H \cup K$, say colored π_v. If there is an edge of W uv such that $\pi_u = \pi_v$, then we would have a monochromatic copy of $(H \cup K) + (H \cup K)$, and hence a monochromatic copy of $G = H + K$, under π', a contradiction. Thus the function $\pi : V(W) \to \{1,\dots,k\}$ defined by $\pi(v) = \pi_v$ is a k-coloring of W, and we are done. $\square$

We do not know whether NP-completeness holds for $(H + K)$ 2-colorability (unless $H + K = K_2$, of course). The family of graphs of the form $H + K$ (or its complement) includes many graphs but, for example, leaves out G being a cycle of length at least 5. We include these in the next result (which starts at $k = 2$, rather than $k = 3$). The proof relies on hypergraph colorings from the previous section, and before we proceed, we will need the following correlation between P colorings and hypergraph colorings [11]. For a graph property P and a graph F, as in [11] we can form a hypergraph H_F^P on the vertices of F such that the edges of H correspond to the subsets of vertices of F that induce a vertex P 2-critical subgraph of F. For $P = -G$, the edges of H_F^P correspond precisely to the induced subgraphs of F isomorphic to G. Moreover, as noted in Theorem 1.1 of [11], $\chi(H_F^P) = \chi(F:P)$.

Proposition 4.3. *Let G be 2-connected. Then* $-G\,\mathrm{col}(k)$ *is* NP-*complete for* $k \geqslant 2$.

Proof. Again, $-G\,\mathrm{col}(k) \in \mathrm{NP}$. Let G have order r. We transform $\mathrm{Hyp}(r,k,r)$ into $-G\mathrm{col}(k)$.

Given an r-uniform hypergraph H of girth greater than r, we can construct a graph F (via the Nešetřil–Rödl construction [32]) by placing a copy of G on each edge of H (i.e., for each edge e of H, we take a fixed bijection f_e from e to $V(G)$ and define a graph edge between $u, v \in e$ if and only if $f_e(u)f_e(v)$ is an edge of G). As H has girth > 2, this is well defined, and in fact, if G' is any 2-connected subgraph of F of order at most r, then from the girth of H being more than r, G' must be contained in some edge of H. In particular, the only copies of G are those induced by the edges of H.

Thus $H = H_F^{-G}$, and $\chi(H) = \chi(F : -G)$. It follows that H is k-colorable iff F is $-G$ k-colorable. As the transformation is polynomial, we are done. $\square$

By considering complementary properties, we can summarize our results as follows.

Theorem 4.4. *The problem of* $-G$ *k-colorability is* NP-*complete for*
- *G or $\overline{G}$ disconnected and $k \geqslant 3$, and*
- *G or $\overline{G}$ 2-connected and $k \geqslant 2$.*

In particular, $-K_m\mathrm{col}(k)$ is NP-complete for $k \geqslant 3$ if $m = 2$ and for $k \geqslant 2$ if $m \geqslant 3$. $-C_m\mathrm{col}(k)$ is NP-complete for $k \geqslant 2$ for all $m \geqslant 3$. As $P_2 = K_2$, $P_3 = K_1 + \overline{K_2}$ and $\overline{P_m}$ is 2-connected for $m \geqslant 5$, we have $-P_m\mathrm{col}(k)$ is NP-complete for (i) $m = 2$, 3 and $k \geqslant 3$, and (ii) $m \geqslant 5$ and $k \geqslant 2$.

One of the most interesting problems is whether $-P_3$ 2-colourability is NP-complete. The results also conspicuously leave out the complexity of $-P_4\mathrm{col}(k)$; in fact, P_4 is the smallest graph that is not one of the forms discussed in Theorem 4.4. We can, however, prove an NP-completeness result for $-P_4$ k-colorability.

Proposition 4.5. $-P_4\mathrm{col}(k)$ *is* NP-*complete for all* $k \geqslant 3$.

Proof. Membership in NP is clear. Let F be a fixed $-P_4$ k-chromatic graph. Such a graph F can be produced via the substitution operation, but note that its order is constant (depending on k, of course). We transform $\mathrm{Gra}(k)$ into $-P_4\mathrm{col}(k)$.

Let G be any graph. We construct a new graph G' by taking a copy of K_3 with a distinguished vertex z and joining z to exactly one vertex in each component of G. As the components of a graph can be found in polynomial time, we can construct G' from G in polynomial time. G' is connected, of order at least 3, and it is easy to verify that for all $k \geqslant 3$, G is k-colorable if and only if G' is. We construct yet another graph

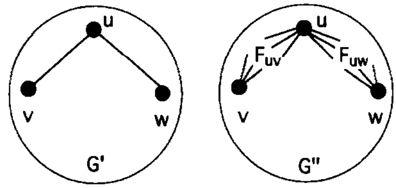

Fig. 2.

G'' from G' by taking a copy F_{uv} of F for each edge uv of G', removing the edge uv from G' and joining u and v to all vertices of F_{uv} (see Fig. 2).

G'' is clearly constructible from G' in polynomial time, so the transformation $G \to G''$ is polynomial. We need to show G is k-colorable if and only if G'' is $-P_4$ k-colorable.

Suppose first G is not k-colorable. It follows that G' is not k-colorable as well. If G'' had a $-P_4$ k-coloring π, then since G' is not k-colorable, for some edge xy of G', $\pi(x) = \pi(y)$. Let $\pi(x) = j$. Now j must be used on some vertex s of F_{xy}, as F_{xy} is not $-P_4$ $(k-1)$-colorable. Since G' is connected and of order at least 4, without loss there is another vertex w of G' such that xw is also an edge of G'. The color j again must appear on some vertex t of F_{xw}, a contradiction as then $\{y,s,x,t\}$ induces a monochromatic copy of P_4. Thus G'' cannot have a $-P_4$ k-coloring.

On the other hand, if G has a k-coloring π, then G' has a k-coloring π'. We can extend π' to a map $\pi'' : V(G'') \to \{1,\ldots,k\}$ by letting the restriction of π'' to G' be π', and the restriction of π'' to any F_{xy} be any $-P_4$ k-coloring of F_{xy}, for all edges uv of G'. Each color class of π'' is the disjoint union of graphs arising by substituting P_4–free graphs into the vertices of a star. Since $-P_4$ is closed under substitution and each star is P_4–free, we conclude that each color class of π'' is P_4-free and hence G'' has a $-P_4$ k-coloring π''.

Thus we have polynomially transformed Gra(k) into $-P_4$col(k), so we conclude that $-P_4$col(k) is NP-complete for all $k \geqslant 3$. $\square$

We remark that Burr [16] investigated the complexity of Ramsey edge colorings of graphs. That is, given fixed graphs G_1 and G_2, what is the complexity of determining whether the edges of a graph F can be colored with 2 colors so that color class i $(i = 1,2)$ does not contain a partial subgraph isomorphic to G_i? This can be viewed as an edge analogue of G-free colorings. Burr proved NP-completeness results for G and H 3-connected, and provided polynomial algorithms when G and H are stars.

5. Remarks

There are many properties of interest for which the complexity of P colorings is not known. It would be useful to have a general result that classified the complexity of P

colorings, but we feel that this is highly unlikely to be found. Along those lines, we can show the following.

Proposition 5.1. *Let P be a property closed under substitution such that every vertex P 2-critical graph is 2-connected. Then $P\mathrm{col}(k)$ is* NP*-hard for all $k \geqslant 3$ (and* NP-*complete if $P\mathrm{col}(1) \in$* NP).

Proof. We find a polynomial transformation of $\mathrm{Gra}(k)$ into $P\mathrm{col}(k)$. Such a transformation will imply the NP-hardness for all $k \geqslant 3$ (and NP-completeness if $P\mathrm{col}(1) \in$ NP).

Let W be a fixed vertex P 2-critical graph with two distinguished vertices x and y. We take a fixed P k-chromatic graph F_k and construct W' by substituting F_k for every vertex of W but x and y. Given a graph G, we construct a graph G' on $V(G)$ by taking, for each edge uv, a copy W'_{uv} of W' and identifying x_{uv} and y_{uv} with u and v in a 1–1 fashion (and thus uv is an edge of W' if and only if xy is an edge of W). This construction is clearly polynomial since the order of W' does not depend on G. We need to show now that G is k-colorable if and only if G' is P k-colorable.

Suppose first that G has a k-coloring π. We extend π to G' by letting the restriction of π to each of the substituted copies of F_k in W'_{uv} (uv and edge of G) be a P k-coloring of it. As every P 2-critical graph is 2-connected, and as P is closed under substitution, we see that each color class of G' is a P graph. Thus G' is P k-colorable.

Conversely, if $\pi : V(G') \to \{1,\ldots,k\}$ is a P k-coloring of G', then for any edge uv of G, each color appears on each copy of F_k in W'_{uv} (since $\chi(F_k : P) > k-1$). It follows that $\pi(u) \neq \pi(v)$, for otherwise the color class $\pi^{-1}(u)$ would contain a copy of W, a vertex P 2-critical graph. Thus the restriction of π to the vertices of G is, in fact, a k-coloring of G, and G is k-colorable. □

As the property of perfection is closed under substitution and every minimally imperfect graph is 2-connected, this yields an alternative proof from [11] that perfect k-colorability is NP-complete for $k \geqslant 3$.

Finally, we mention that if we extend the graphs we consider from finite to infinite graphs, then we can prove undecidability results for P k-colorability analogous to the ones in the previous section. As in [15], we define a countable graph F to be *doubly periodic* if the following conditions hold. The vertices of F are $a_{i,j,l}$, where $i,\ j \in \mathbb{N}$ and $l \in \{0,\ldots,L\}$ (L a fixed positive integer). All the edges of F are within a cell $A_{i,j} = \{a_{i,j,l} : 0 \leqslant l \leqslant L\}$ or between cells whose i and j coordinates differ by at most one. Moreover, the adjacency of $a_{i,j,l}$ and $a_{i',j',l'}$ depends only on $i-i'$, $j-j'$, l and l'. Burr noted that such an F is really a finitary object (i.e. has a finite description) so no problems with recursion arise. Burr proved the following result:

Theorem 5.2 (Burr's Theorem [15]). *Let $k \geqslant 3$ be a fixed integer. Then we can construct a doubly periodic graph W such that graph k-colorability is undecidable* *for the class of graphs formed from W by adding in a finite set of vertices and edges.*

Using this result, we can prove analogs to the results of the previous section. For example:

Theorem 5.3. *Let $k \geqslant 3$ be a fixed integer, and let H and K be fixed nonempty finite graphs. Then we can construct a doubly periodic graph W' such that $-(H+K)\mathrm{col}(k)$ is undecidable for the class of graphs formed from W' by adding in a finite set of vertices and edges.*

Proof. As in the proof of Proposition 4.2, let F be a fixed finite graph with clique number $\omega(H \cup K)$ that is not $-(H \cup K)$ k-colorable. We form W' by substituting a copy of F for every vertex of the graph W in Burr's Theorem. It is not hard to see that W' is doubly periodic as well. Moreover, if G is any graph formed from W by adding a finite number of vertices and edges, then G', the graph formed by substituting a copy of F for each vertex of G, arises also by adding a finite number of vertices and edges to W'.

By the same argument as in the proof of Proposition 4.2, G is k-colorable if and only if G' is $-(H \cup K)$ k-colorable. Note that the construction $G \to G'$ is clearly computable (i.e. recursive). Thus if there is an algorithm to check whether the graphs G' are $-(H \cup K)$ k-colorable, then there would be an algorithm to solve graph k-colorability on the class of graphs in Burr's Theorem, a contradiction. Therefore, the problem of determining whether a graph formed from W' by adding a finite number of vertices and edges is $-(H \cup K)$ k-colorable is undecidable. □

From this theorem, we can easily derive the undecidability of $-(H \cup K)$ k-colorability ($k \geqslant 3$) for a suitable recursive family of graphs. Other constructions (for both hypergraphs colorings and graph P colorings) of the previous two sections will generate similar undecidability results, and illustrate the general observation that 'infinite' analogs of NP-complete problems often turn out to be undecidable.

We conclude by mentioning the most interesting open problems. First of all, what is the complexity of $-P_3$ 2-colorability? Note that a graph is P_3-free if and only if it is the disjoint union of complete graphs. Thus a P_3 2-coloring of a graph is equivalent to a partition of its vertices into two sets, each inducing a disjoint union of complete graphs. This is tantalizingly close to graph 2-colorability, but perhaps sufficiently different to be NP.

The classes of graphs left out by Theorem 4.4 are those for which it and its complement have vertex connectivity 1. This, of course is a small family of graphs, and perhaps its structure can be utilized for showing at least that $-G$ k-colorability is NP-complete for any graph G of order at least 2 and any $k \geqslant 3$.

Finally, general NP-hardness results are of interest for specific properties. For example, is perfection 2-colorability NP-hard?

Acknowledgements

Most of the material presented in this paper has also appeared in the author's doctoral dissertation [8]. The author wishes to thank Derek Corneil for his insightful comments and suggestions. The author also acknowledges the Natural Sciences and Engineering Research Council of Canada for financial assistance.

References

[1] M.O. Albertson, R.E. Jamison, S.T. Hedetniemi and S.C. Locke, The subchromatic number of a graph, Discrete Math. 74 (1989) 33–49 .

[2] G. Benadé, I. Broere and J.I. Brown, A construction of uniquely C_4-free colourable graphs, Quaestiones Math. 13 (1990) 259–264.

[3] C. Berge, Graphs and Hypergraphs (North Holland, New York, 1979).

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, New York, 1976).

[5] I. Broere and C. Mynhardt, Generalized colorings of outerplanar and planar graphs, in: Graph Theory with Applications to Algorithms and Computer Science (Wiley, New York, 1985) 151–161.

[6] M. Browiecki and P. Mihók, On (n,P)-partitionable graphs, in: Graphs, Hypergraphs and Applications (Teubner, Leipzig, 1985) 15–18.

[7] M. Browiecki and P. Mihók, Hereditary properties of graphs, in: Advances in Graph Theory (Teubner, Vishwa, 1991) 41–68.

[8] J.I. Brown, A theory of generalized graph colourings, Ph. D. Thesis, Department of Mathematics, University of Toronto (1987).

[9] J.I. Brown and D.G. Corneil, On generalized graph colorings, J. Graph Theory 11 (1987) 87–99.

[10] J.I. Brown and D.G. Corneil, Perfect colourings, Ars Combin 30 (1990) 141–159.

[11] J.I. Brown and D.G. Corneil, Graph properties and hypergraph colourings, Discrete Math. 98 (1991) 81–93.

[12] J.I. Brown and D.G. Corneil, On uniquely $-G$ k-colourable graphs, Quaestiones Math. 15 (1992) 477–487.

[13] J.I. Brown and V. Rödl, A new construction of k-folkman graphs, Ars Combin. 29 (1990) 265–269.

[14] J.I. Brown and V. Rödl, A ramsey type problem concerning vertex colourings, J. Combin Theory. (B) 52 (1991) 45–52.

[15] S.A. Burr, Some undecidable problems involving edge-colorings and vertex-colorings of graphs, Discrete Math. 50 (1984) 171–177.

[16] S.A. Burr, On the computational complexity of Ramsey-type problems, in: Mathematics of Ramsey Theory (Springer, New York, 1990) 46–52.

[17] E.J. Cockayne, Colour classes for r-graphs, Canad. Math. Bull. 15 (1972) 349–354.

[18] C.J. Colbourn, M.J. Colbourn, K.T. Phelps and V. Rödl, Colouring Steiner quadruple systems, Discrete Appl. Math. 4 (1982) 103–111.

[19] P. Erdös and A. Hajnal, On chromatic numbers of graphs and set systems, Acta Math. Acad. Sci. Hungar. 17 (1966) 61–99.

[20] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, SIAM J. Appl. Math. 18 (1970) 19–24.

[21] M. Frick and M.A. Henning, Extremal results on defective colorings of graphs, Discrete Math. 126 (1994), 151–158.

[22] M.R. Garey and D.S. Johnson, Computers and Intractibility, (W.H. Freeman, New York, 1979).

[23] J. Gimbel, D. Kratsch and L. Stewart, On cocolourings and cochromatic numbers of graphs, Discrete Appl. Math. 48 (1994) 111–127.

[24] F. Harary, Conditional colorability of graphs, in: F. Harary and J. Maybee, eds., Graphs and Applications, Proceedings of the 1st Col. Symp. Graph Theory (Wiley, New York, 1985) 127–136.

[25] F. Harary, J. Nešetřil and V. Rödl, Generalized ramsey theory for graphs XIV: induced ramsey numbers, in: M. Fielder, ed., Graphs and Other Combinatorial Topics (Teubner, Leipzig, 1983) 90–100.

[26] S. Hedetniemi, On partitioning planar graphs, Canad. Math. Bull. 11 (1968) 203–211.
[27] P. Hell and J. Nešetřil, On the complexity of H-colouring, J. Combin. Theory Ser. B 48 (1990) 92–110.
[28] P. Komjath and V. Rödl, Coloring of universal graphs, Discrete Math. 63 (1986) 55–60.
[29] L. Lovász, On chromatic number of finite set systems, Acta Math. Acad. Sci. Hungar. 19 (1968) 59–67.
[30] L. Lovász, Coverings and colorings of hypergraphs, *Proceedings of the 45th Southeastern Conf. Combinatorics, Graph Theory and Computing* (1973) 3–12.
[31] C.M. Mynhardt and I. Broere, Generalized colourings of graphs, in: Y. Alavi et al, ed., Graph Theory with Applications to Algorithms and Computer Science (Wiley, New York, 1985) 583–594.
[32] J. Nešetřil and V. Rödl, Partitions of vertices, Comm. Math. Univ. Carolinae 17 (1976) 85–95.
[33] J. Nešetřil and V. Rödl, The ramsey property for graphs with forbidden complete subgraphs, J. Combin Theory Ser. B 20 (1976) 243–249.
[34] J. Nešetřil and V. Rödl, On ramsey graphs without cycles of short odd length, Comm. Math. Univ. Carolinae 20 (1979) 565–582.
[35] K.T. Phelps and V. Rödl, On the algorithmic complexity of coloring simple hypergraphs and steiner triple systems, Combinatorica 4 (1984) 79–88.
[36] N. Robertson and P.D. Seymour, Graph minors. XX. Wagner's conjecture, manuscript.
[37] E. Sampathkumar, Generalizations of independence and chromatic numbers of a graph, Discrete Math. 115 (1993) 245–251.
[38] E. Sampathkumar and C.V. Venkatachalam, Chromatic partitions of a graph, Discrete Math. 74 (1989) 227–239.
[39] E.R. Scheinerman and A. Trenk, On generalized perfect graphs: bounded degree and bounded edge perfection, Discrete Appl. Math. 44 (1993) 233–245.
[40] B. Toft, Colour-critical graphs and hypergraphs, J. Combin Theory Ser. B 16 (1974) 145–161.